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ALMOST FIXED POINT THEOREMS FOR THE EUCLIDEAN PLANE

MATHEMATICS

BY

J. DE GROOT, H. DE VRIES AND T. VAN DER WALT [1])

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1. *Introduction*

For many non-compact spaces, e.g. the Euclidean plane, the fixed point property (f.p.p.) does not hold. However, there might be a substitute which in the compact case is equivalent to the fixed point property.

Definition. Let X be a topological space, F a family of mappings of X into itself and Ω a family of finite coverings of X. Then X is said to have the *almost fixed point property (a.f.p.p.) with respect to F and Ω* if, for every $f \in F$ and every $\alpha \in \Omega$, there exists a member $U \in \alpha$ such that $U \cap f[U] \neq \phi$; in other words, there is a point $p \in X$ such that p and $f(p)$ belong to the same member U of α.

If X is a compact Hausdorff space, then X has the f.p.p. if and only if X has the a.f.p.p. with respect to continuous mappings and finite open coverings. (The "only if" part is clear; the "if" part is established by an easy argument.)

It can be shown that the Euclidean space E^n has the a.f.p.p. with respect to continuous mappings and finite coverings by open sets with compact boundaries. This means that any continuous mapping of E^n into itself either has a fixed point or else there are points near infinity for which the images also are near infinity, e.g. a translation.

Instead of turning to questions of a general nature, we prove three theorems for E^2. In the first theorem we prove that E^2 has the a.f.p.p. with respect to (arbitrary) continuous functions and finite coverings by *convex* open sets. In the second theorem we consider coverings by sets of a more general nature; in fact, we can take any finite covering by means of arcwise connected sets, but the mappings are restricted to *orientation preserving isometries*. Even then, the proof needs careful reasoning and an example found by the third author will show – among other things in section 4 – that there is no corresponding theorem for orientation reversing isometries. These results will also be presented in the third author's thesis [5].

It should be observed that already HOPF [3] (see also WOLFF and

[1]) The third author received a grant from the South African Council for Scientific and Industrial Research.

DENJOY [6]) obtained results like the following (stated in different terminology): A unicoherent topological space has the a.f.p.p. with respect to continuous mappings and coverings of order two by closed connected sets.

2. Theorem 1. *The Euclidean plane E^2 has the a.f.p.p. with respect to continuous mappings and finite coverings by convex open sets.*

Remarks. 1. It is easy to see that a corresponding theorem does not hold for infinite (convex open) coverings.

2. It should be possible to generalize theorem 1 by replacing E^2 by E^n.

We shall use the following lemma (with $n=2$) in the proof of theorem 1.

Lemma 1. (FORT [2].) Let d be a positive number and let $B^n=\{x \in E^n \mid \|x\|<d\}$. Let $f : B^n \to B^n$ be continuous. Then for each $\varepsilon>0$ there exists a point $x \in B^n$ such that $\|x-f(x)\|<\varepsilon$.

Proof. Let $\varepsilon>0$ be given. We may obviously assume that $\varepsilon<d$. Let $C^n=\{x \in B^n \mid \|x\| \leqslant d-\varepsilon\}$, and define a retraction $r : B^n \to C^n$ by

$$r(x)=\begin{cases} (d-\varepsilon)x/\|x\|, & \text{for } x \in B^n \setminus C^n \\ x, & \text{for } x \in C^n. \end{cases}$$

Then $rf \mid C^n \to C^n$ is continuous and according to the Brouwer fixed point theorem for the n-cell, there exists a point $c \in C^n$ such that $rf(c)=c$. Since $\|r(x)-x\|<\varepsilon$ for all $x \in B^n$, we have $\|c-f(c)\|=\|rf(c)-f(c)\|<\varepsilon$.

Definition. A *strip* is the closure of a connected open set in E^2, whose boundary consists of two (different) parallel straight lines. Let P be a strip bounded by the lines L_1 and L_2, and let L_3 be a (closed) segment perpendicular to L_1 and L_2 which connects a point of L_1 with a point of L_2. Then the closure of a component of $P \setminus L_3$ is called a *half-strip*. The segment L_3 is called the *base* of the half-strips, and the lines [rays] bounding a strip [half-strip] are called the *sides* of the strip [half-strip].

It is easy to verify that a convex subset K of E^2 with interior points has the following properties:

(i) If K° (the interior of K) contains a line, then it contains a strip.

(ii) If K° contains a ray, then it contains a half-strip.

Proof of theorem 1. Let $f : E^2 \to E^2$ be a continuous mapping and $\alpha=\{U_i\}_{i=1}^n$ a finite covering of E^2 by convex open sets. We may assume that E^2 does not belong to α. Since α is a finite covering and E^2 is unbounded, there exist pairs of different members of α which have an unbounded intersection. Such an intersection satisfies either (i) or (ii) above, and we choose, if possible, a strip in each unbounded intersection; otherwise, we choose a half-strip. Divide each strip into two half-strips, such that the intersection of the ensuing half-strips is their common base. Let $P_1, P_2, \ldots, P_k$ be the collection of half-strips. We may choose them such that $P_i \cap P_j$ $(i \neq j)$ is bounded, and we shall suppose that this was done. Further, we choose an open disk B_1 such that the following conditions are fulfilled:

(i) If $U_i \cap U_j$ is bounded, then $\overline{U_i \cap U_j} \subset B_1$.

(ii) $P_i \cap P_j \subset B_1$ $(i \neq j;\ i, j = 1, 2, \ldots, k)$.

(iii) B_1 contains the bases of the half-strips as well as the points of intersection of the prolongations of the sides of the half-strips.

Let B_2 be an open disk, concentric with B_1 and such that $\bar{B}_1 \subset B_2$. We shall now construct a homeomorphism $\varphi : E^2 \to B_2$, such that $\{\varphi[U_i]\}_{i=1}^{n}$ can be extended to an open covering of $\bar{B}_2$.

We shall assume that the collection of half-strips is cyclically ordered by the positive orientation of the boundary of B_2, and that this ordering is given by $P_1, P_2, \ldots, P_k$ "modulo k". We also assign an order to the sides of each P_i $(i \equiv 1, 2, \ldots, k)$: if we traverse the boundary of B_2 in positive direction, then we pass from the "first side" of P_i to its "second side".

Let S_i denote the closure of that component of $E^2 \backslash (B_1 \cup P_1 \cup \ldots \cup P_k)$ which lies between the second side of P_i and the first side of P_{i+1} $(i \equiv 1, 2, \ldots, k)$. P_i and S_i are thus constructed so that there exists a member $U_{j(i)} \in \alpha$ with the property that

(iv) $P_i \cup S_i \cup P_{i+1} \subset U_{j(i)} \qquad (i \equiv 1, 2, \ldots, k)$.

We are now ready to define the homeomorphism $\varphi : E^2 \to B_2$. It will be done in such a way that $P_i \backslash B_1$ is contracted onto $P_i \cap (B_2 \backslash B_1)$, and S_i onto $S_i \cap (B_2 \backslash B_1)$ $(i \equiv 1, 2, \ldots, k)$, while $\bar{B}_1$ is mapped identically onto itself.

$z \in P_i \backslash B_1$ $(i \equiv 1, 2, \ldots, k)$: Let $L_i(z)$ be the line through z parallel to the sides of P_i, and let $r_i(z) = \text{dist}\ (z, L_i(z) \cap bd(B_1))$, where $bd(B_1)$ denotes the boundary of B_1. Define $f_i(z)$ to be the point which divides the segment $L_i(z) \cap (B_2 \backslash B_1)$ in the ratio $r_i(z) : 1 + r_i(z)$. It is easy to verify that f_i is a continuous one-to-one mapping of $P_i \backslash B_1$ onto $P_i \cap (B_2 \backslash B_1)$, and that its inverse is continuous.

$z \in S_i$ $(i \equiv 1, 2, \ldots, k)$: Let a_i be the point in which the prolongation of the second side of P_i intersects the prolongation of the first side of P_{i+1}, and let $\overline{a_i z}$ be the closed segment connecting a_i and z. Let $s_i(z) = \text{dist}\ (z, \overline{a_i z} \cap bd(B_1))$, and define $g_i(z)$ to be the point which divides the segment $\overline{a_i z} \cap (B_2 \backslash B_1)$ in the ratio $s_i(z) : 1 + s_i(z)$. Then g_i is a continuous one-to-one mapping of S_i onto $S_i \cap (B_2 \backslash B_1)$, and its inverse is continuous. (If P_i and P_{i+1} are parallel, then we define g_i in the same way as f_i was defined.)

$z \in \bar{B}_1$: Let $h : \bar{B}_1 \to \bar{B}_1$ be the identity mapping.

Any two of the functions f_i, g_i and h coincide on the intersection of their (closed) domains of definition and hence φ, defined by

$$\varphi(z) = \begin{cases} f_i(z) & (z \in P_i \backslash B_1; \quad i = 1, 2, \ldots, k), \\ g_i(z) & (z \in S_i; \quad i = 1, 2, \ldots, k), \\ z & (z \in \bar{B}_1), \end{cases}$$

is a continuous mapping of E^2 onto B_2. Similarly, φ^{-1} is well-defined and continuous; hence φ is a homeomorphism.

For each $U_i \in \alpha$, let $U_i' = \varphi[U_i]$ and let $\varphi(\alpha) = \{U_i'\}_{i=1}^n$. For each $U_{j(i)}$ satisfying (iv), let $V_{j(i)} = U_j'{}_{(i)} \cup ((P_i \cup S_i \cup P_{i+1}) \cap bd(B_2))$. It is easily seen that the $V_{j(i)}$, together with the remaining U_i', form an open covering of $\bar{B}_2$. Denote this covering by $\beta = \{W_i\}_{i=1}^m$.

Let $f' = \varphi f \varphi^{-1}$. Then $f' : B_2 \to B_2$ is continuous and according to lemma 1, for each positive integer n, there exists a point $y_n \in B_2$ such that $\|y_n - f'(y_n)\| < 1/n$. Let τ be the Lebesgue number of $\bar{B}_2$ with respect to β, and choose n such that $1/n < \tau$. According to the lemma of Lebesgue, there exists a set $W_k \in \beta$ such that y_n, $f'(y_n) \in W_k$. But y_n, $f'(y_n) \in B_2$, so that y_n and $f'(y_n)$ lie in the same member of $\varphi(\alpha)$. Hence, if x_n is that point of E^2 for which $\varphi(x_n) = y_n$, then x_n and $f(x_n)$ lie in the same member of α.

3. If the mappings are restricted to translations, we can require less of the covering sets to obtain a theorem similar to theorem 1: "convex open" may then be replaced by "arcwise connected".

We shall need the following two lemmas.

Lemma 2. Let $X_1, X_2, \ldots, X_n$ be sets, let $X = \bigcup_{i=1}^{n} X_i$ and let $f : X \to X$ be a mapping. Then there exists a set X_i and a positive number k $(1 \leqslant i, k \leqslant n)$ such that $X_i \cap f^k[X_i] \neq \phi$.

Proof. For each $x \in X$, at least two of the $n+1$ elements $x, f(x), \ldots, f^n(x)$ belong to one and the same set X_i; say $f^r(x), f^s(x) \in X_i$ $(0 \leqslant r < s \leqslant n)$. Then $f^s(x) \in X_i \cap f^{s-r}[X_i]$.

Lemma 3. Let A be an arcwise connected subset of E^2, and let $f : E^2 \to E^2$ be a translation, such that there exists a positive integer k with $A \cap f^k[A] \neq \phi$. Then $A \cap f[A] \neq \phi$ too.

Proof. Let f be given by $f(x) = x + a$, for all $x \in E^2$, where $a \in E^2$ is a fixed vector. We may suppose that the positive X-axis has the same direction as a. Let k be the smallest positive integer such that $A \cap f^k[A] \neq \phi$. Suppose $k > 1$. We are going to derive a contradiction. There exists a point $b \in A$ such that $b + ka \in A$ also, and we can find an arc J, contained in A, which connects b and $b + ka$. Let

$$P = \{(x, y) \in J \mid (u, v) \in J \Rightarrow y \geqslant v\}, \text{ and}$$
$$Q = \{(x, y) \in J \mid (u, v) \in J \Rightarrow y \leqslant v\}.$$

Since J is compact, $P \neq \phi$ and $Q \neq \phi$. (P and Q contain respectively the "upper extreme" and "lower extreme" points of J.) Since $J \cap f[J] = \phi$, J is not a segment, and since it is compact, we can find a point $p \in P$ and a point $q \in Q$, such that, if J_1 is the part of J which connects p and q (including p and q), then $J_1 \cap P = \{p\}$, $J_1 \cap Q = \{q\}$, and $p \neq q$.

Let L_1 and L_2 be straight lines parallel to the X-axis, passing through p and q respectively, and let S be the strip determined by these lines. J_1 separates S into two disjoint sets, each of which is simply connected and both open and closed in S. The same holds for the images of J_1 under the interates of f.

Since $J_1 \cap f[J] = \phi$ and $f[J]$ is connected, any two points of $f[J]$, in particular $b+a$ and $q+a$, lie in the same part of S with respect to the separation by J_1. Since f is a translation, $b+ka$ and $q+ka$ lie in the same part of S with respect to the separation by $f^{k-1}[J_1]$. Since $q+(k-2)a$ and $q+ka$ lie in different parts of S with respect to this separation, $b+ka$ and $q+(k-2)a$ lie in different parts. Also, q and $q+(k-2)a$ lie in the same part of S with respect to this separation, and hence q and $b+ka$ lie in different parts. But q and $b+ka$ are connected by J, and $J \subset S$, so that $J \cap f^{k-1}[J_1] \neq \phi$, implying that $A \cap f^{k-1}[A] \neq \phi$, in contradiction with the choice of k.

Definition. Let X be a topological space. Two continuous mappings $f, g : X \to X$ are said to be *topologically equivalent* if there exists a homeomorphism h of X onto itself such that $f = h^{-1}gh$. If X is a metric space, then a mapping $f : X \to X$ is called a *topological isometry* if it is topologically equivalent to a distance preserving mapping of X into itself.

In the case of the plane we have the following criterion for a mapping to be a topological translation (Sperner [4]): A mapping $f : E^2 \to E^2$ is topologically equivalent to a translation if and only if f is an orientation preserving homeomorphism such that, for each set $G \subset E^2$, which is the closure of a bounded domain and whose boundary is a Jordan curve, there exists a positive integer N such that $G \cap f^n[G] = \phi$ for all integers n with $|n| \geqslant N$.

We now state and prove

Theorem 2. *The Euclidean plane has the a.f.p.p. with respect to orientation preserving topological isometries and finite coverings by arcwise connected sets.*

Proof. It is a well-known result that an orientation preserving isometry of the Euclidean plane is topologically equivalent either to a rotation or to a translation. In the first case there is a fixed point, and in the second case theorem 2 immediately follows from lemmas 2 and 3.

Corollary. The Euclidean plane has the a.f.p.p. with respect to orientation preserving topological isometries and finite coverings by connected open sets.

For, a connected open subset of a Euclidean space is arcwise connected.

An example orally communicated by Professor R. D. Anderson shows that theorem 2 cannot be extended to higher dimensions: There is a covering α of E^3 by four non-empty connected open sets and a topological translation $f : E^3 \to E^3$ such that $U \cap f[U] = \phi$ for all $U \in \alpha$.

4. We recall the following

Definition. A connected topological space X is *unicoherent* if, whenever $X = A \cup B$, $A \neq \phi$, $B \neq \phi$, with both A and B closed and connected in X, it follows that $A \cap B$ is connected.

A connected topological space trivially has the a.f.p.p. with respect to arbitrary mappings and coverings consisting of two connected open sets. A unicoherent topological space has the a.f.p.p. with respect to continuous mappings and coverings consisting of three connected open sets. Before showing this, we prove the following

Lemma 4. Let X be a unicoherent topological space and $\alpha = \{U, V, W\}$ a covering of X by three non-empty connected open sets. Then, if $U \cap V \cap W = \phi$, α has two disjoint members.

Proof. Suppose, on the contrary, that $U \cap V \neq \phi$, $U \cap W \neq \phi$ and $V \cap W \neq \phi$. Then

$$X = U \cup (V \cup W) \qquad \text{(connected summands)},$$

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W) \quad \text{(connected summands), and}$$

$$(U \cap V) \cap (U \cap W) = U \cap V \cap W = \phi,$$

contradicting the unicoherence of X.

Theorem 3. A unicoherent topological space X has the a.f.p.p. with respect to continuous mappings and coverings consisting of three connected open sets.

Proof. Let $f : X \to X$ be a continuous mapping and $\alpha = \{U, V, W\}$ a covering of X by three connected open sets. We may suppose that the empty set does not belong to α, and that $U \cap V \cap W = \phi$. Let U and V be the disjoint members of α given by lemma 4. Then $U \cap W \neq \phi$, $V \cap W \neq \phi$, since X is connected. Suppose that $W \cap f[W] = \phi$. Since $f[W]$ is connected and $U \cap V = \phi$, either $f[W] \subset U$ or $f[W] \subset V$. In either case the theorem is proved, e.g., if $f[W] \subset U$ then $f[U \cap W] \subset f[W] \subset U$ and hence $U \cap f[U] \neq \phi$.

Corollary. E^n has the a.f.p.p. with respect to continuous mappings and coverings consisting of three connected open sets.

For, E^n is unicoherent (Borsuk [1]).

The question arises whether a unicoherent topological space has the a.f.p.p. with respect to continuous mappings and coverings consisting of four (or more) connected open sets. Further, can "orientation preserving" be omitted from the hypotheses of theorem 2?

Both these questions are answered negatively by the following example, in which we have a covering of E^2 by four connected open sets U_1, U_2, U_3, U_4 and a transflection f (i.e., a reflection followed by a translation in the direction of the axis of reflection) such that $U_i \cap f[U_i]=\phi$ $(i=1, 2, 3, 4)$.

Let

$$\begin{aligned}
V &= \{(x, y) \in E^2 \mid 0<x<1, \ -1\leq y<1\},\\
r(x, y) &= (x, y)+(2, 0), \text{ for all } (x, y) \in E^2,\\
s(x, y) &= (x, y)+(\tfrac{2}{3}, 0), \text{ for all } (x, y) \in E^2,\\
W &= \{(x, y) \in E^2 \mid y< -1\},\\
V_1 &= \bigcup_{n=-\infty}^{\infty} r^n[V], \quad U_1 = V_1 \cup W,\\
U_2 &= s[U_1], \qquad U_3 = s[U_2],\\
U_4 &= \{(x, y) \in E^2 \mid y > 0\}.
\end{aligned}$$

The transflection f is defined as follows:

$$\begin{aligned}
u(x, y) &= (x, -y), &&\text{for all } (x, y) \in E^2,\\
t(x, y) &= (x, y)+(1, 0), &&\text{for all } (x, y) \in E^2,\\
f &= tu.
\end{aligned}$$

It is easy to verify that $U_i \cap f[U_i]=\phi$ $(i=1, 2, 3, 4)$. Note that f reverses the orientation, and that each of the intersections $U_i \cap U_j$ $(i\neq j)$ has countably infinitely many components.

Problems. 1. Does the Euclidean plane have the a.f.p.p. with respect to orientation preserving homeomorphisms onto and finite coverings by connected open sets?

2. Does the Euclidean plane have the a.f.p.p. with respect to continuous mappings and finite coverings by connected open sets such that the intersection of each pair of members of the covering has only a finite number of components?

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